

ex: If A is a square matrix then T is one-to-one if and only if it's onto

lecture 12

Eigenvalues and Eigen Vectors

Let A be an $n \times n$ matrix. The scalar λ is called an eigen value of A if there is a non zero vector x such that $Ax = \lambda x$

eigen
vector

eigen
value

$$\text{Ex } A = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}, \quad x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$Ax_1 = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$Ax_1 = 2x_1$$

eigen value = 2

$$x_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$Ax_2 = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$Ax_2 = -1x_2$$

eigen value = -1

$$\text{Ex } A = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad x_1 = \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}$$

$$Ax_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}$$

$$Ax_1 = 0x_1$$

eigen value = 0

How to find the eigen values of the eigen vectors?

$$A\underline{x} = \lambda \underline{x}$$

$$= \lambda I \underline{x}$$

$$\lambda I \underline{x} - A \underline{x} = 0$$

$$(\lambda I - A) \underline{x} = 0$$

Det $(\lambda I - A)$ must be zero so that \underline{x} is not trivial and λ has ^{many} values.

Ex. Find the eigen values and corresponding eigen vectors of $A = \begin{pmatrix} 2 & -12 \\ 1 & -5 \end{pmatrix}$

$$\det(\lambda I - A) = 0$$

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 2 & -12 \\ 1 & -5 \end{pmatrix} = \begin{pmatrix} \lambda - 2 & 12 \\ -1 & \lambda + 5 \end{pmatrix} \rightarrow \text{characteristic equation}$$

$$\det = \lambda^2 + 3\lambda + 2 = 0 \quad (\lambda + 1)(\lambda + 2) = 0$$

$$\boxed{\lambda = -1, -2} \rightarrow \text{eigen values}$$

for $\lambda = -1$

$$\begin{pmatrix} -3 & 12 \\ -1 & 4 \end{pmatrix} \underline{x} = 0 \quad \begin{pmatrix} -1 & 4 \\ -3 & 12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} -1 & 4 \\ -3 & 12 \end{pmatrix} \xrightarrow{R_2 - 3R_1} \begin{pmatrix} -1 & 4 \\ 0 & 0 \end{pmatrix} \quad \text{let } x_2 = t$$

$$-x_1 + 4x_2 = 0 \quad x_1 = 4t$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t \begin{pmatrix} 4 \\ 1 \end{pmatrix} \rightarrow \text{eigen vector}$$

for $\lambda = -2$

$$\begin{pmatrix} -4 & 12 \\ -1 & 3 \end{pmatrix} \xrightarrow[\text{form}]{\text{echelon}} \begin{pmatrix} -1 & 3 \\ 0 & 0 \end{pmatrix} \quad \text{let } x_2 = s$$

$$x_1 = 3s$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = s \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad \text{characteristic equation } \det(\lambda I - A) = 0$$

$$\det \begin{pmatrix} \lambda-1 & -3 & 0 \\ -3 & \lambda-1 & 0 \\ 0 & 0 & \lambda+2 \end{pmatrix} = (\lambda+2) \begin{pmatrix} \lambda-1 & -3 \\ -3 & \lambda-1 \end{pmatrix}$$

to find characteristic equation easily ~~not~~ subtract entries in main diagonal from λ and change the signs of the remaining entries

$$\det = (\lambda+2)(\lambda^2 - 2\lambda - 8) = (\lambda+2)(\lambda+2)(\lambda-4) = 0 \quad \lambda = -2, 4$$

for $\lambda = -2$

$$\begin{pmatrix} -3 & -3 & 0 \\ -3 & -3 & 0 \\ 0 & 0 & -1 \end{pmatrix} \xrightarrow[\text{echelon form}]{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} -3 & -3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \begin{array}{l} \text{let } x_2 = t, x_3 = s \\ \Rightarrow -3x_1 - 3x_2 = 0 \\ x_1 = -t \end{array}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

\Rightarrow eigen vectors

for $\lambda = 4$

$$\begin{pmatrix} 3 & -3 & 0 \\ -3 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix} \xrightarrow[\text{form}]{\text{echelon}} \begin{pmatrix} 3 & -3 & 0 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{l} \text{let } x_1 = t \\ x_2 = t \end{array} \quad 6x_3 = 0 \quad x_3 = 0$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Special case (triangular matrix)

Theorem: If A is an $n \times n$ ~~not~~ triangular matrix, then its eigen values are the entries on its main diagonal

$$\text{eg } A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \lambda = 2, 2, 2$$

Diagonalisation

Similar Matrices: Two $m \times n$ matrices A and B are called similar if $B = P^{-1}AP$ holds for some invertible matrix P .

- If A and B are similar, they must have

1. Same determinant
2. Same rank
3. Same eigen values
4. Same trace

* Trace is the sum of all entries in main diagonal.

Diagonalizable matrix: An $m \times n$ matrix A is diagonalizable if A is similar to a diagonal matrix

$$P^{-1}AP = \text{diagonal matrix}$$

Steps for diagonalising an $m \times n$ matrix

1. Find n linearly independent eigen vectors p_1, p_2, \dots, p_n for the matrix A with corresponding eigen values $\lambda_1, \lambda_2, \dots, \lambda_n$. If n linearly independent eigen vectors do not exist then A is not diagonalizable
2. Let P be the $m \times n$ matrix whose columns consists of these eigen vectors
3. The diagonal matrix $D = P^{-1}AP$ will have the eigen values $\lambda_1, \lambda_2, \dots, \lambda_n$ on its main diagonal (and zeros elsewhere)

$$\text{eg. } A = \begin{pmatrix} 3 & 2 & 1 \\ 5 & 8 & 0 \\ 0 & 2 & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \text{eigen vectors}$$

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 2 & 1 & 1 \end{pmatrix}$$

Show that matrix A is diagonalizable then find a matrix P such that $P^{-1}AP$ is diagonal

$$A = \begin{pmatrix} -1 & -1 & 1 \\ 3 & 1 & -1 \\ -3 & 1 & -1 \end{pmatrix}$$

characteristic equation

$$\det \begin{pmatrix} \lambda - (-1) & -1 & 1 \\ -1 & \lambda - 1 & -1 \\ 3 & -1 & \lambda - (-1) \end{pmatrix} = 0 \Rightarrow \lambda^3 - 3\lambda^2 - 4\lambda + 12 = 0$$

$$(\lambda - 3)(\lambda - 2)(\lambda + 2) = 0$$

$$\lambda = 3, 2, -2$$

Since the roots are different \rightarrow A is diagonalizable

$\lambda = 3$

$$\begin{pmatrix} 2 & -1 & 1 \\ -1 & 0 & -1 \\ 3 & -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

echelon form

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{let } x_3 = t \quad x_1 = -t \quad x_2 = t$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$\lambda = 2$

$$\begin{pmatrix} 1 & -1 & 1 \\ -1 & -1 & -1 \\ 3 & -1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$\lambda = -2$

$$\begin{pmatrix} -3 & -1 & 1 \\ -1 & -3 & -1 \\ 3 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ 1 \end{pmatrix} = \frac{t}{4} \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$$

2- $P =$
~~27~~ $\begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 4 \end{pmatrix}$ or $(P|I) \Rightarrow (I|P^{-1})$
 $P^{-1} = \frac{1}{|P|} \text{adj}(P)$

$P^{-1} = \begin{pmatrix} 0 & -1 & -1 \\ \frac{1}{5} & \frac{1}{5} & 0 \\ \frac{1}{5} & \frac{1}{5} & 1 \end{pmatrix}$

$P^{-1}AP = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

Ex. Diagonalize the matrix $A = \begin{pmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{pmatrix}$

Characteristic equation

$\det \begin{pmatrix} \lambda-5 & -8 & -16 \\ -4 & \lambda-1 & -8 \\ 4 & 4 & \lambda+11 \end{pmatrix} = 0 \quad (\lambda+3)^2(\lambda-1) = 0 \quad \lambda = -3, -3, 1$

$\lambda = -3$

$\begin{pmatrix} -8 & -8 & -16 \\ -4 & -4 & -8 \\ 4 & 4 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

$\lambda = 1$

$\begin{pmatrix} -4 & -8 & -16 \\ -4 & 0 & -8 \\ 4 & 4 & 12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$

$P = \begin{pmatrix} -1 & -2 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \quad P^{-1} = \begin{pmatrix} -1 & 0 & -2 \\ 1 & 1 & 3 \\ -1 & -1 & -2 \end{pmatrix}$

$$\text{Aug. } \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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END!